FOURIER SPACE SOLUTION OF BURGERS' EQUATION

Roman Shcherbakov
Applied Mathematics 201A Final Project
4, January, 2007

1. Introduction

There exist, of course, an analytic solution of Burgers equation

$$u_t + u \cdot u_x = \nu u_{xx} \tag{1}$$

with initial condition $u(x,0) = u_0(x)$. Nevertheless, many authors still solve it numerically [Muhammad I. Bhatti et al 2006], [Kurganov, Tadmor, 2000] in order to

- reveal and use some specific properties of Burgers' equation,
- test numerical techniques that are then used to solve more sophisticated equations.

In my study I propose a new effective technique of numerical solution of viscous Burgers' equation with high Reynolds numbers. (It's "new" so far as I haven't found anything similar in the Internet). The approach addresses the incentives described above. It's quite fast, designed to explicitly treats shocks and will be used in application to other equations with stiff or discontinuous behavior, primarily hydrodynamic equations.

The paper comprises several sections. Description of the method is given in section 2. Comparison to the exact solution is made in section 3. The results are followed by the discussion of possible value of the method in section 4.

2. Numerical technique

Let us for the sake of simplicity consider initial conditions that have the following properties:

- are odd,
- exponentially decay at infinity,
- give rise to shock early in simulation,
- shock decays later in simulation.

Therefore, spectrum of the solution

$$u(k,t) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} u(x,t)\sin(kx)dx \tag{2}$$

is also odd, linearly grows at small k, is inversely proportional to k for intermediate k as a consequence of shock and exponentially decays for large k because of finite size of shock region for viscous equation.

Such a shape can be approximated by Pade approximant P_{2N}^{2N-1} with an exponential cut-off

$$u(k,t) \approx \frac{\sum_{n=1}^{N} b_n \cdot k^{2n-1}}{1+\sum_{n=1}^{N} c_n \cdot k^{2n}} \exp(-s \cdot k^2). \tag{3}$$

Because the separation of scales is present, the procedure can be easily done despite the excessive number of parameters for constructing the approximant from the series expansion

$$u(k,t_0) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{k^n}{n!} \int_0^{+\infty} u(x,t_0) \frac{d\sin(kx)}{dk}|_{k=0} dx. \tag{4}$$

One may use the iterative procedure. One need to find P_{2N}^{2N-1} from the expansion of $u(k,t)$ and then fit transition of the spectrum to the exponential cut-off. After doing this one may correct the expansion with value of the exponent found and repeat derivation of the approximant.

For small N it is easier, however, to use "Nonlinearfit" package in Mathematica.

So far, I constructed the initial condition (3) in Fourier space, but the equation itself (1) still remains in the real space. What I want to do next is to make all the coefficients functions of time $b_n(t), c_n(t), s(t)$ and write $2N+1$ equations on them. Anyhow, I need to use the values of $u(x,t)$ in the shock zone along with the values outside the shock zone making the equations sensitive for both linear and inverse linear parts of the spectrum.

The approximant (3) can also be written in form

$$u(k,t) \approx \exp(-s(t) \cdot k^2) \left(\sum_{n=1}^{N} \frac{d_n(t) \cdot k}{1+e_n(t) \cdot k^2} \right). \tag{3'}$$

Inverse Fourier transform

$$\tilde{u}(x,t) = \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} u(k,t) \sin(kx) dk \tag{5}$$

can be done analytically for each term of the sum (3') only either at $x=0$ or at very large x far from the shock region where I can neglect the exponential cut-off (this unobvious statement is proved numerically, see Fig.10). Moreover, it is possible to analytically find all the derivatives at $x=0$ of inverse Fourier transform in this case. Conservation of mass and momentum are very tempting to use. Unfortunately, these quantities can only be approximated in case of (3').

I make somewhat obvious choice of the equations: $2N$ equations are the values of the first $2N$ non-zero x derivatives of (1) + one equation is (1) taken at large x, where the approximant fits the spectrum. I get $2N+1$ non-linear ordinary differential equations for $2N+1$ functions $b_n(t), c_n(t), s(t), n = 1..N$ with some initial conditions. Solving the equations I obtain the spectrum at any given time. Inverse Fourier transform should be calculated to get the function in the real space.

3. Example

Let me calculate time evolution of the solution for Burgers' equation with the initial condition $u(x,0) = -5 \cdot 1.1^{-x^2/2} \tanh(1.1x)$ and viscosity $\nu = 0.05$ that corresponds to $Re \sim 20$. Its spectrum is shown on Fig 1. The method described in the previous section should work for virtually all the spectra satisfying the posed constraints, but the better fit with the same number of terms is achieved for the initial condition that already has the shock. This initial condition gives rise to the shock. For the solution at time $t = 1$ and its fit with the approximant with $N = 1$ see Fig 2. We have three unknowns $b_1(t), c_t(t), s(t)$ for $N = 1$. Corresponding three equations are the first and the third derivatives of (1) at $x = 0$ and (1) at $x = 7$.

I start at $t = 1$ and integrate to $t = 2$. Because the initial fit at $t = 1$ is quite good, one may expect my approximate solution to be in a good agreement with the exact solution at some later time say $t = 2$. Spectrum of the exact solution and my approximant at $t = 2$ are shown on Fig. 3. Corresponding real space solutions are shown in Fig. 4 and Fig. 5. Coefficients $b_1(t)$ (Fig. 6) and $s(t)$ (Fig. 7) responsible correspondingly for the linear stage and for the cut-off show very good agreement with the exact solution, much better than one might expect from the 1-st order approximant. The value of $c_1(t)$ (Fig. 8), however, slightly overshoots the expected value, making the spectrum slightly lower and the shock wave in the real space smaller in the amplitude.

4. Discussion

Let me check first of all, whether my way of calculating the solution has numerical difficulties in its realization. Fitting of the initial conditions doesn't seem to be a problem. Neither is the derivation of equations. Transformation from (3) to (3') could be inaccurate if roots of denominator and zeros of numerator of P_{2N}^{2N-1} are crowded near the origin of coordinates, near the positive ray of the real axis or near each other. I plotted those points for P_8^7. Fig. 9 shows that only the proximity of roots to each other may be a problem. I haven't proven the convergence of the algorithm. Initial fit seems to converge exponentially in N for approximant (3) not requiring many terms to get a reasonable fit like in [Muhammad et al., 2006]. Equations on its coefficients don't seem to have any exponential instability on the error in the initial conditions, therefore, algorithm should work.

The proposed method can easily be adapted for hydrodynamic equations and extended to non-symmetric configurations and moving shocks, therefore, it is applicable to virtually any equation with discontinuous behavior.

Why is this technique so good? The equations even for $N = 1$ are quite large and non-linear, however, the solution is described by analytic non-steep functions. It means that we don't have any scheme-independent constraint on the Courant number in numerical scheme, the scourge of even nowaday computational methods like [Kurganov, Tadmor, 2000] and [Amundsen, Bruno 2005]. The absence of such a constraint has a "physical" explanation: we are now able to predict the pattern of the solution in the whole, without any "unexpected" shock, appearing in a random place.

So, large system of equations may actually take less computational time to solve than equation (1) with shock discontinuities. Taking time step larger than prescribed by constraint on the Courant number causes exponential growth of the discrepancy from the exact solution near discontinuity. That discrepancy is called Gibbs phenomenon. Gibbs phenomenon can be overcome using Fourier-Pade transform [Driscoll, Fornberg, 2001]. Idea of these authors can possibly be fruitful for the development of another technique for breaking the limitation of the time step compared to the spatial step.

The author is willing to continue working in this direction and is wondering about the publicability of the work done so far.

5. Bibliography

1. Muhammad I. Bhatti1, Dambaru D. Bhatta, "Numerical solutions of Burgers' equation in a B-polynomial basis", 2006, Phys. Scr., v. 73, p. 539

2. Driscoll, T.A., Fornberg B.; "A Pade-based algorithm for overcoming the Gibbs phenomenon", 2001, Numerical Algorithms

3. Baker; "Pade approximants", 1995, Cambridge University Press

4. Amundsen, D.E., Bruno O.; "Time Stepping Via One-Dimensional Pade Approximation", 2005

5. Kurganov, Tadmor; "New high-resolution central schemes for nonlinear conservation laws and convection-diffusion equations", 2000

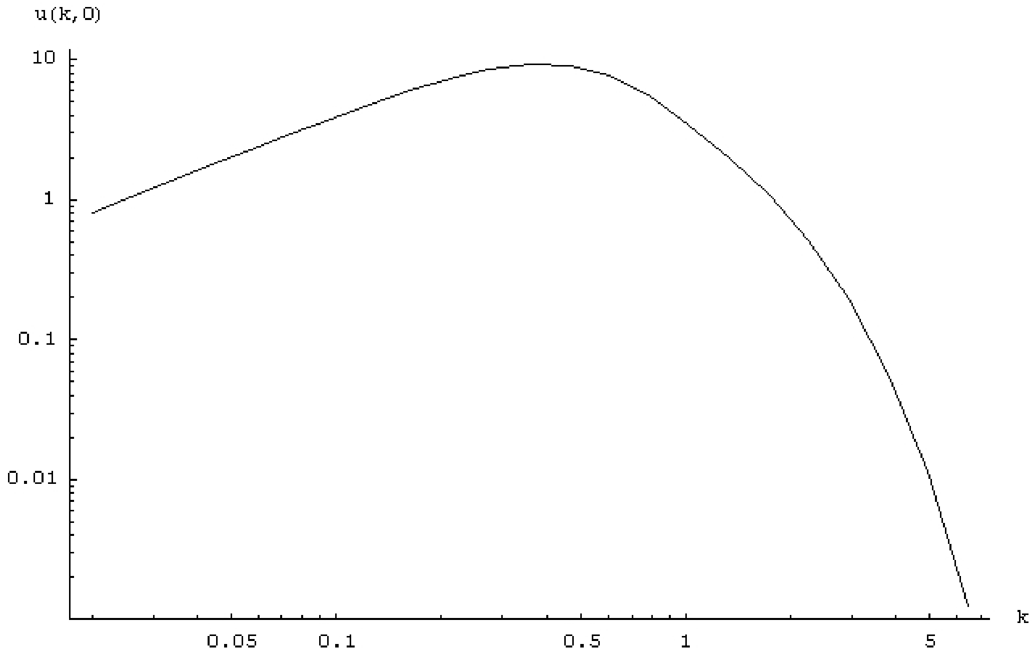

FIGURE 1. Spectrum of the initial condition $u(x, 0) = -5 \cdot 1.1^{-x^2/2} \tanh(1.1x)$.

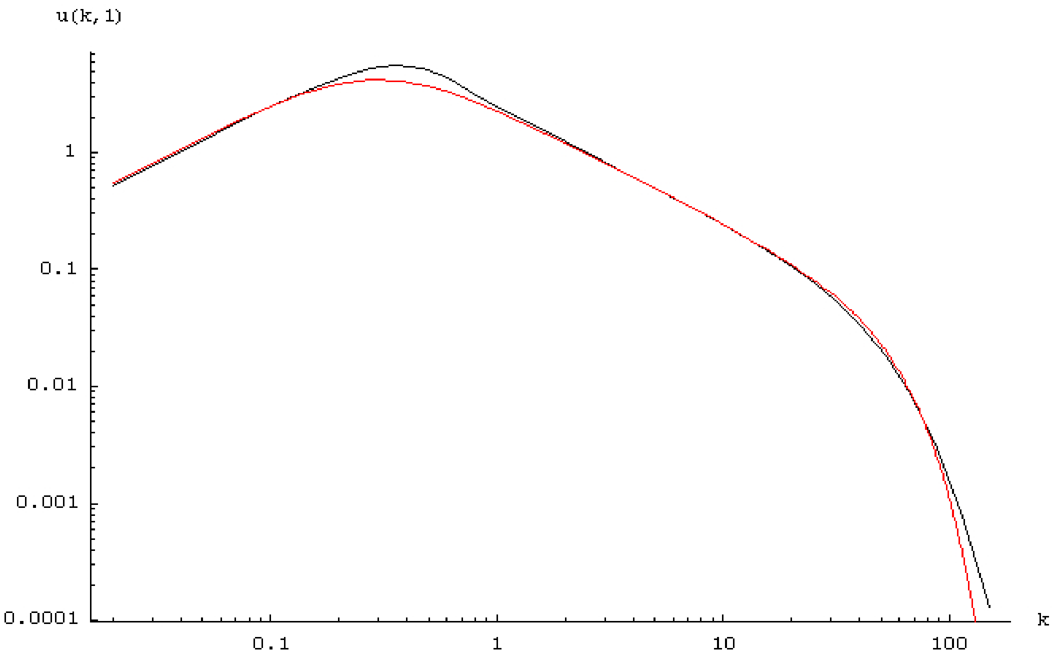

FIGURE 2. Spectrum of the solution at $t = 1$. Black line is the accurate solution, red line is the fit, initial condition for the described method.

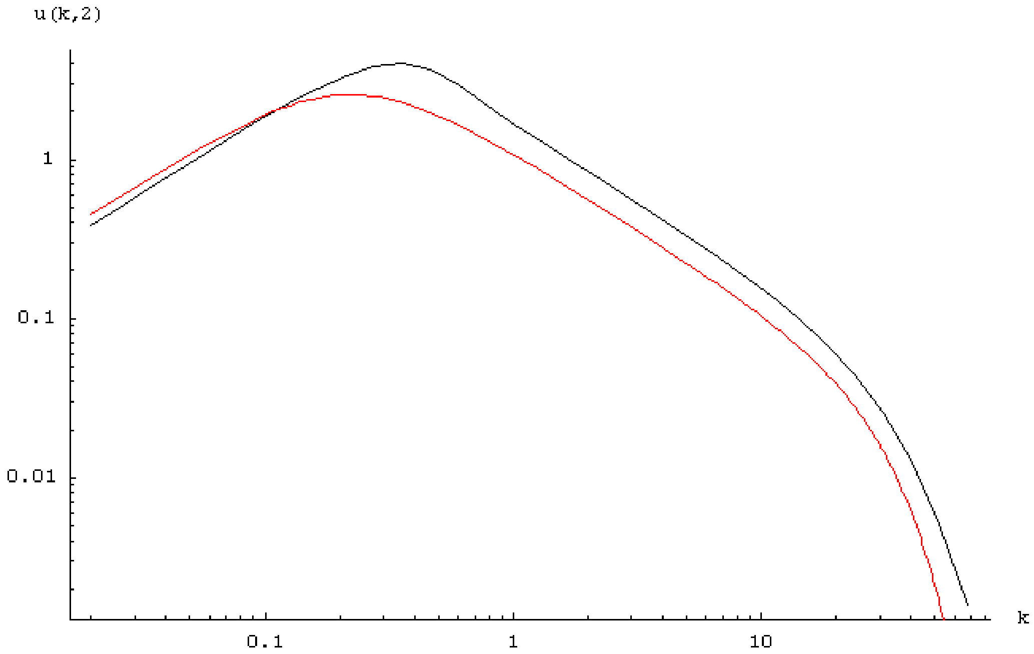

FIGURE 3. Spectrum of the solution of $t = 2$. Black line is the accurate solution, red line is the solution, calculated using my algorithm.

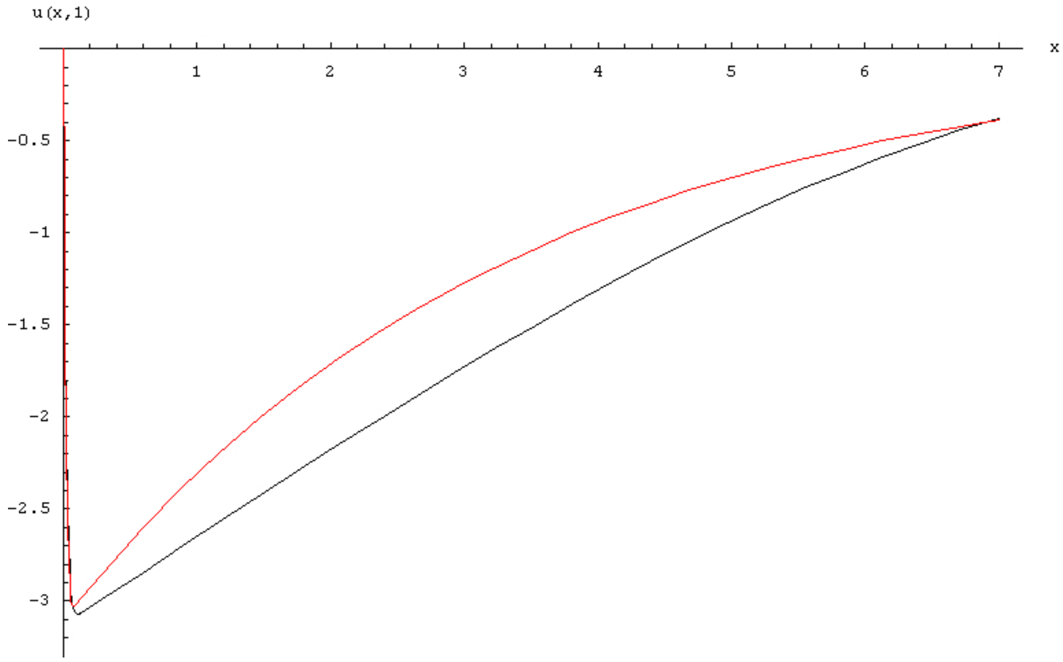

FIGURE 4. Accurate solution (black line) and inverse Fourier spectrum of the approximant (red line) at $t = 1$.

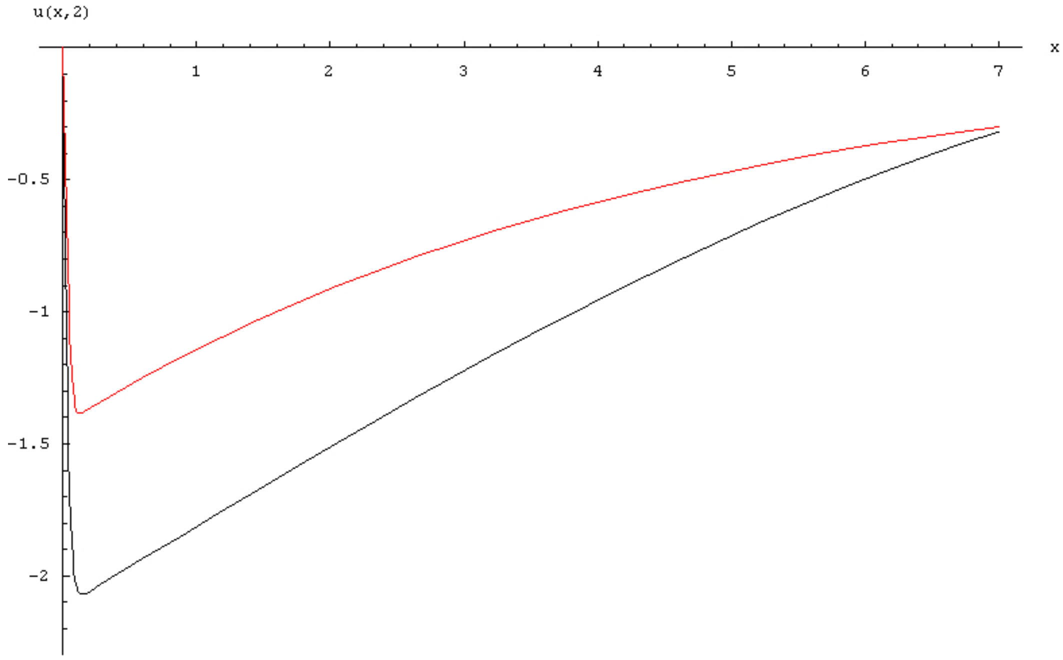

FIGURE 5. Accurate solution (black line) and inverse Fourier spectrum of the calculated approximant (red line) at $t = 2$.

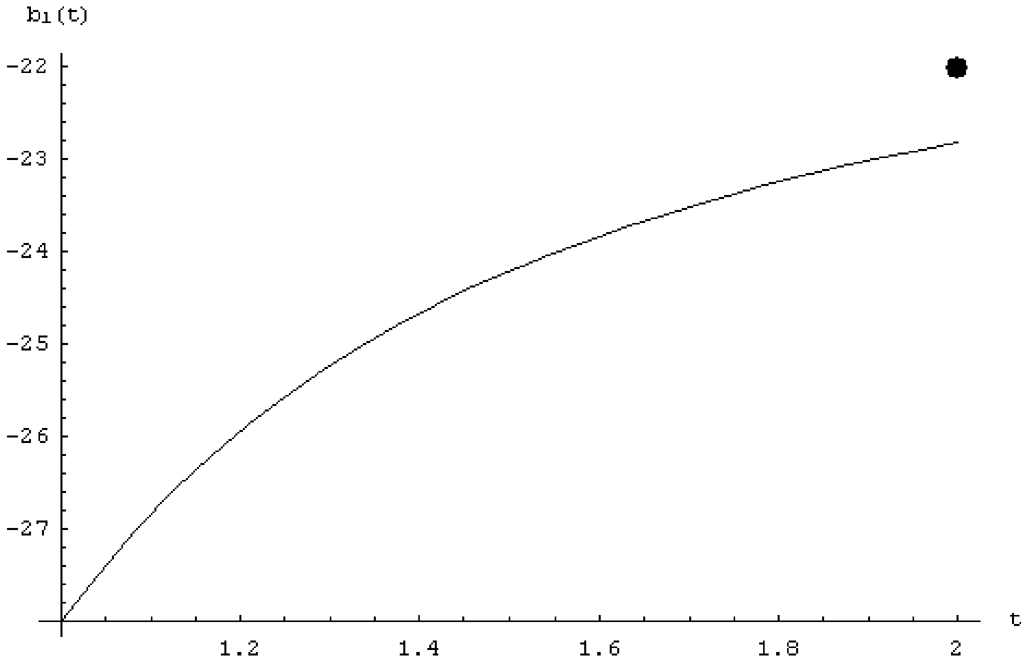

FIGURE 6. Predicted time evolution of the coefficient $b_1(t)$, responsible for the linear part of the spectrum. Black dot is the fit to the Fourier transform of the accurate solution at $t = 2$.

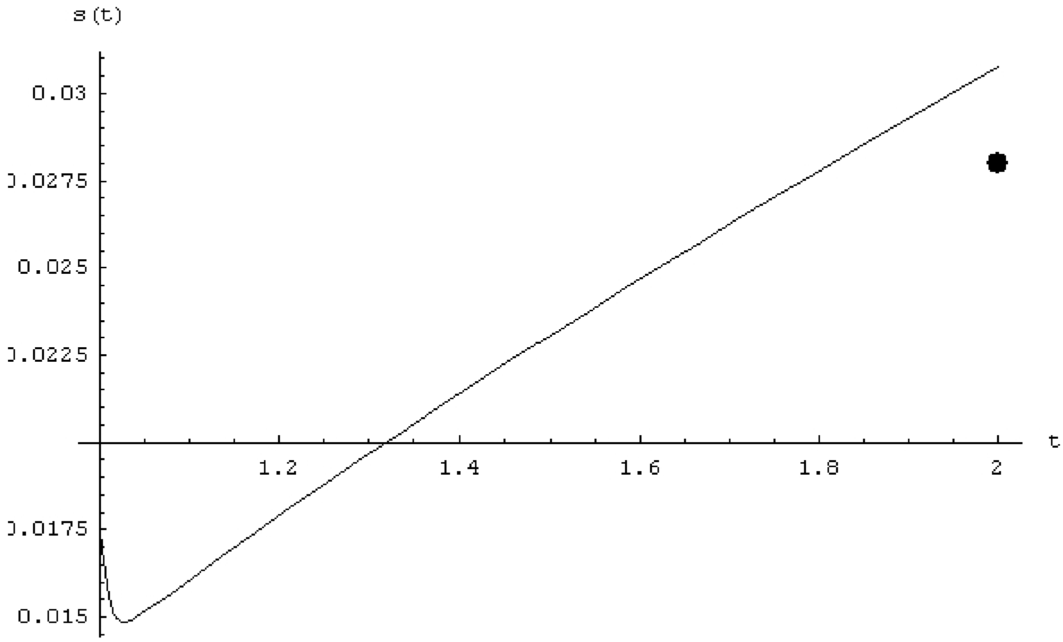

FIGURE 7. Predicted time evolution of the coefficient $s(t)$, responsible for the exponential cut-off of the spectrum. Black dot is the fit to the Fourier transform of the accurate solution at $t = 2$. Small hook in the beginning of integration must be due to inaccurate initial conditions.

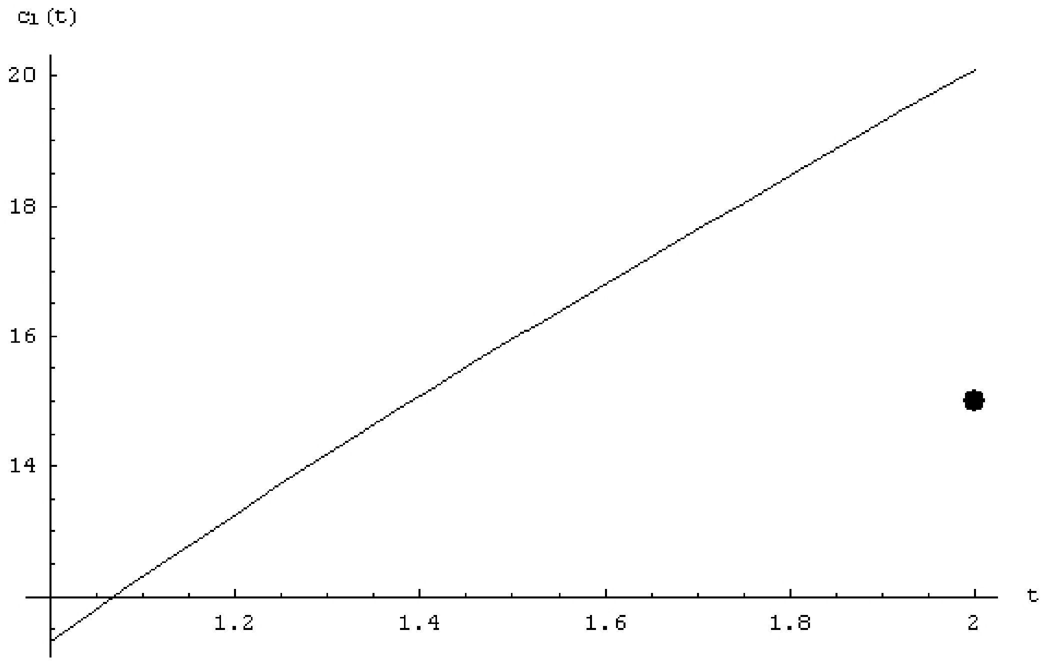

FIGURE 8. Predicted time evolution of the coefficient $c_1(t)$, responsible for the shock part of the spectrum. Black dot is the fit to the Fourier transform of the accurate solution at $t = 2$.

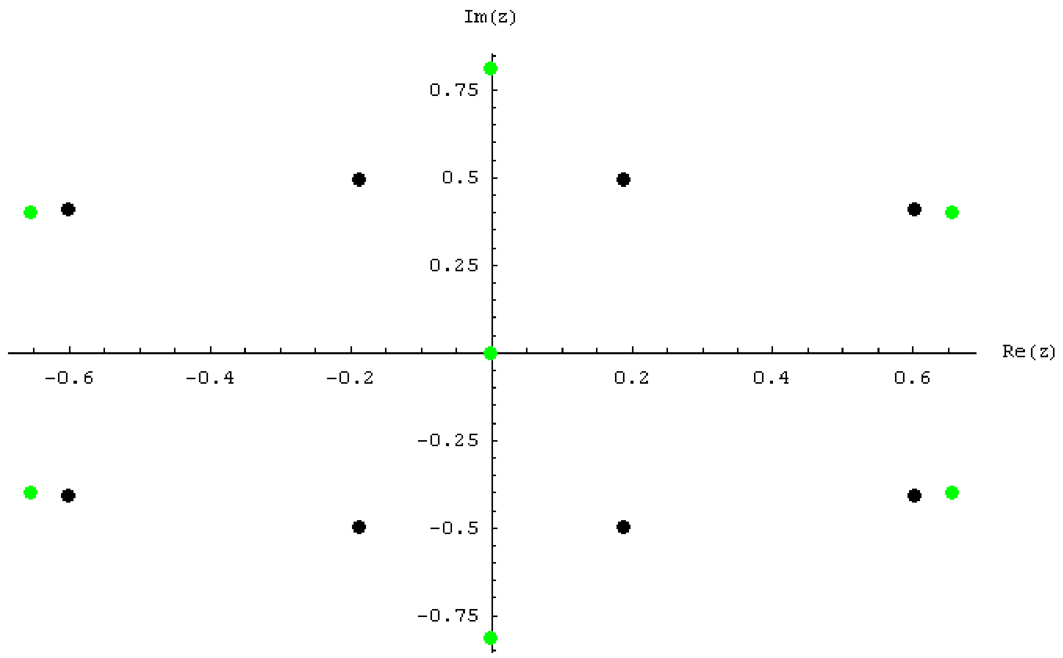

FIGURE 9. The distribution of roots of the denominator(black) and zeros of the numerator(green) of Pade approximant P_8^7, part of the fit (3).

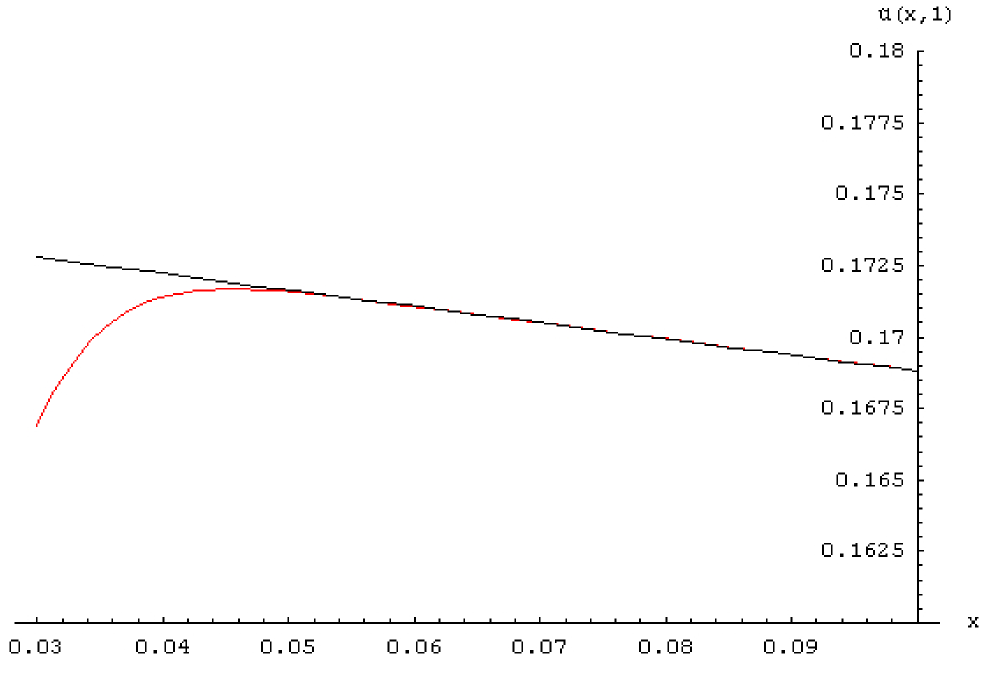

FIGURE 10. Comparison of values of $\int_0^{+\infty} \frac{b_1 k}{1+c_1 k^2} \exp(-s \cdot k^2) \sin(kx) dk$ (black line) and $\int_0^{+\infty} \frac{b_1 k}{1+c_1 k^2} \sin(kx) dk$ (red line) for the values of coefficients b_1, c_1, s from the fit at $t = 1$.