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PROPAGATION EFFECTS IN MAGNETIZED TRANSRELATIVISTIC PLASMAS

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ABSTRACT

The transfer of polarized radiation in magnetized and non-magnetized relativistic plasmas is an area of research with numerous flaws and gaps. The present paper is aimed at filling some gaps and eliminating the flaws. Starting from a Trubnikov's linear response tensor for a vacuum wave with $\mathbf{k}=\omega/c$ in thermal plasma, the analytic expression for the dielectric tensor is found in the limit of high frequencies. The Faraday rotation and Faraday conversion measures are computed in their first orders in the ratio of the cyclotron frequency Ω_0 to the observed frequency ω. The computed temperature dependencies of propagation effects bridge the known non-relativistic and ultra-relativistic limiting formulas. The fitting expressions are found for high temperatures, where the higher orders in Ω_0/ω cannot be neglected. The plasma eigenmodes are found to become linearly polarized at much larger temperatures than thought before. The results are applied to the diagnostics of the hot ISM, hot accretion flows, and jets.

Subject headings: radiative transfer — polarization — magnetic fields

1. INTRODUCTION

We learn much of our information about astrophysical objects by observing the light they emit. Observations of the polarization properties of light can tell us the geometry of the emitter, strength of the magnetic field, density of plasma, and temperature. The proper and correct theory of optical activity is essential for making accurate predictions. While the low-temperature propagation characteristics of plasma are well-established (Landau & Lifshits 1980), the theory of relativistic effects has not been fully studied. In this paper I discuss the propagation effects through a homogeneous magnetized relativistic plasma. A non-magnetized case emerges as a limit of the magnetized case. The discussion is divided into three separate topics.

Two linear plasma propagation effects are Faraday rotation and Faraday conversion (Azzam & Bashara 1987). Traditionally, these effects are considered in their lowest orders in the ratio β of the cyclotron frequency Ω_0 to the circular frequency of light ω, id est in a high-frequency approximation. The distribution of particles is taken to be thermal

$$dN=\frac{n\exp(-\gamma/T)}{4\pi m^3c^3TK_2(T^{-1})}d^3p \qquad (1)$$

with the dimensionless temperature T in the units of particle rest mass temperature mc^2/k_B. The Faraday rotation measure RM and conversion measure are known in a non-relativistic $T\ll 1$ and an ultra-relativistic $T\gg 1$ limits (Melrose 1997c). I derive a surprisingly simple analytic expression for arbitrary temperature T.

The smallness of $\beta=\Omega_0/\omega$, $\beta\ll 1$ in the real systems led some authors (Melrose 1997a) to conclude that the high-frequency approximation will always work. However, there is a clear indication that it breaks down at high temperatures $T\gg 1$. It was claimed that the eigenmodes of plasma are linearly polarized for high temperatures $T\gg 1$ (Melrose 1997c), because the second order term $\sim\beta^2$ becomes larger than the first order term $\sim\beta$ due to the T dependence. The arbitrarily large T-factor may stand in front of higher order expansion terms in β of the relevant expressions. I find the generalized rotation measure as a function of β and T without expanding in β and compare the results with the known high-frequency expressions. The high-T behavior of the plasma response is indeed significantly different.

Plasma physics involves complicated calculations. This led to a number of errors in the literature (Melrose 1997c), some of which have still not been fixed. In the article I check all the limiting cases numerically and analytically and expound all the steps of derivations. Thus I correct the relevant errors and misinterpretations made by previous authors, hopefully not making new mistakes. The analytical and numerical results are obtained in Mathematica 6 system. It has an enormous potential in these problems (Marichev 2008).

The paper is organized as follows. The formalism of plasma response and calculations are described in §2. Several applications to observations can be found in §3. I conclude in §4 with a short summary and future prospects.

2. CALCULATIONS

2.1. *Geometry of the problem*

I assume the traditional geometry depicted on Figure 1:

- Euclidean basis ($\tilde{\mathbf{e}}^1$, $\tilde{\mathbf{e}}^2$, $\tilde{\mathbf{e}}^3$),
- magnetic field along the third axis $\tilde{\mathbf{B}}=(0,0,B)^T$,
- a wave vector of the wave $\tilde{\mathbf{k}}=k(\sin\theta,0,\cos\theta)^T$ with an angle θ between $\tilde{\mathbf{k}}$ and $\tilde{\mathbf{B}}$.

The basis is rotated from ($\tilde{\mathbf{e}}^1$, $\tilde{\mathbf{e}}^2$, $\tilde{\mathbf{e}}^3$) to ($\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$), so that the wave propagates along $\mathbf{k}=(0,0,k)^T$ in the new basis. The transformation has the form

$$\mathbf{e}^1=\tilde{\mathbf{e}}^1\cos\theta-\tilde{\mathbf{e}}^3\sin\theta,\quad \mathbf{e}^2=\tilde{\mathbf{e}}^2,\quad \mathbf{e}^3=\tilde{\mathbf{e}}^1\sin\theta+\tilde{\mathbf{e}}^3\cos\theta, \qquad (2)$$

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which can be conveniently written as

$$\mathbf{e}^\mu = \tilde{\mathbf{e}}^\nu S^{\nu\mu}, \quad S^{\nu\mu} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}. \quad (3)$$

Vectors and tensors then rotate according to

$$A^\mu = (S^T)^{\mu\nu}\tilde{A}^\nu, \quad \alpha^{\mu\nu} = (S^T)^{\mu\sigma}\tilde{\alpha}^{\sigma\delta}S^{\delta\nu}. \quad (4)$$

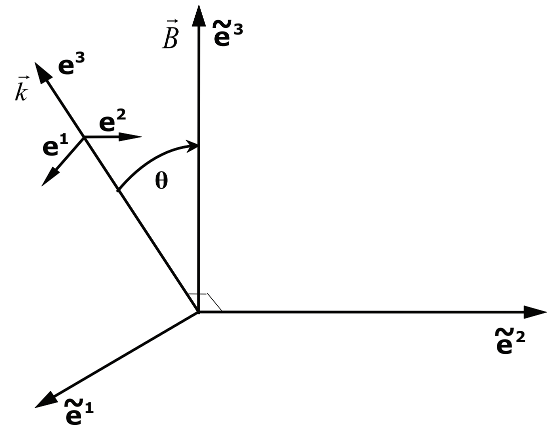

FIG. 1.— Geometry of the problem.

2.2. *Linear plasma response*

The propagation of weak electromagnetic (EM) waves in a homogeneous magnetized plasma can be fully described by the response tensor $\alpha^{\mu\nu}$. It expresses the linear proportionality between the induced current density and the vector potential $j^\mu(\omega) = \alpha^\mu{}_\nu A^\nu(\omega)$. The spatial projection of such defined 4-D tensor $\alpha^\mu{}_\nu$ is equal to the 3-D tensor α_{ij} defined by $\mathbf{j} = \alpha_{ij}\mathbf{A}$.

I consider Trubnikov's form of the response tensor (Trubnikov 1958; Melrose 1997a). I work in a low-density regime, where the plasma response is calculated for a vacuum wave with $|\mathbf{k}| = \omega/c$. I take the tensor $\tilde{\alpha}^{\mu\nu}$ from the first-hand derivations (Trubnikov 1958; Melrose 1997a), make the transformation (4), and take the 1-st and 2-nd components in both indices. Thus the projection onto the $(\mathbf{e}^1, \mathbf{e}^2)$ plane in CGS units is

$$\alpha^\mu{}_\nu(k) = \frac{iq^2 n\omega\rho^2}{cmK_2(\rho)} \int_0^\infty d\xi \left[t^\mu{}_\nu \frac{K_2(r)}{r^2} - R^\mu \bar{R}_\nu \frac{K_3(r)}{r^3} \right], \quad (5)$$

$$t^\mu{}_\nu = \begin{pmatrix} \cos^2\theta\cos\Omega_0\xi + \sin^2\theta & \eta\cos\theta\sin\Omega_0\xi \\ -\eta\cos\theta\sin\Omega_0\xi & \cos\Omega_0\xi \end{pmatrix}, \quad (6)$$

$$R^\mu = \frac{\omega\sin\theta}{\Omega_0}\left(\cos\theta(\sin\Omega_0\xi - \Omega_0\xi), -\eta(1-\cos\Omega_0\xi)\right), \quad (7)$$

$$\bar{R}_\nu = \frac{\omega\sin\theta}{\Omega_0}\left(\cos\theta(\sin\Omega_0\xi - \Omega_0\xi), \eta(1-\cos\Omega_0\xi)\right), \quad (8)$$

and

$$r = \left[\rho^2 - 2i\omega\xi\rho + \frac{\omega^2\sin^2\theta}{\Omega_0^2}\left(2 - \Omega_0^2\xi^2 - 2\cos\Omega_0\xi\right)\right]^{1/2}, \quad (9)$$

where η is the sign of the charge, $K_n(r)$ is the n-th Bessel function of the second kind[1]. The quantity ρ is the dimensionless inverse temperature,

$$\rho = T^{-1} = \frac{mc^2}{k_B T_p}, \quad (10)$$

where T_p the actual temperature of particles. The response of plasma is usually characterized by the dielectric tensor. Its projection onto the $(\mathbf{e}^1, \mathbf{e}^2)$ plane is

$$\varepsilon^\mu{}_\nu = \delta^\mu{}_\nu + \frac{4\pi c}{\omega^2}\alpha^\mu{}_\nu. \quad (11)$$

The wave equation for transverse waves in terms of $\varepsilon^\mu{}_\nu$ is

$$(n_r^2\delta^\mu{}_\nu - \varepsilon^\mu{}_\nu)\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = 0, \quad (12)$$

where E_1 and E_2 are the components of the electric field along $\mathbf{e}^1$ and $\mathbf{e}^2$ and $n_r^2 = k^2c^2/\omega^2$ (Swanson 2003).

2.3. *High frequency limit*

Let me first calculate the limiting expression for $\alpha^\mu{}_\nu$ in the high-frequency limit $\Omega_0 \ll \omega$. I denote

$$\alpha = \omega\xi, \qquad \beta = \frac{\Omega_0}{\omega}, \quad (13)$$

substitute the definitions (13) into the expression (5), and expand the response tensor $\alpha^\mu{}_\nu$ in β. I retain only up to the 2-nd order of the expansion, which gives the conventional generalized Faraday rotation (Melrose 1997c). The first terms of the series of r, $t^\mu{}_\nu$, and $R^\mu\bar{R}_\nu$ read

$$r^2 = r_0^2 + \delta r^2, \quad r_0^2 = \rho^2 - 2i\alpha\rho, \quad \delta r^2 = -\frac{\sin^2\theta}{12}\beta^2\alpha^4, \quad (14)$$

$$t^\mu{}_\nu = \begin{pmatrix} 1 - \cos^2\theta\cdot\alpha^2\beta^2/2 & \alpha\beta\eta\cos\theta \\ -\alpha\beta\eta\cos\theta & 1 - \alpha^2\beta^2/2 \end{pmatrix}, \quad (15)$$

$$R^\mu\bar{R}_\nu = -\frac{\alpha^4\beta^2}{4}\sin^2\theta\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (16)$$

Melrose (1997c) used the approximation $r_0^2 = -2i\alpha\rho$ instead of the expansion (14) and obtained the approximate high-T expressions as his final answers.

However, one can take the emergent integrals, if one considers the exact expansions (14,15,16). Three terms appear in the expanded expression for $\alpha^\mu{}_\nu$:

$$\int_0^\infty d\alpha \left[t^\mu{}_\nu \frac{K_2(r_0)}{r_0^2} \right], \quad (17)$$

$$\int_0^\infty d\alpha \left[t^\mu{}_\nu \frac{K_3(r_0)\delta r^2}{r_0^3} \right], \quad (18)$$

$$\int_0^\infty d\alpha \left[R^\mu\bar{R}_\nu \frac{K_3(r_0)}{r_0^3} \right]. \quad (19)$$

The 2-nd term (18) originates from the expansion of $K_2(r)/r^2$ in r^2 to the first order

$$\frac{K_2(r)}{r^2} - \frac{K_2(r_0)}{r_0^2} = -\frac{\delta r^2}{2}\frac{K_3(r_0)}{r_0^3}. \quad (20)$$

[1] Note that the analogous expression in Melrose (1997c) has an extra factor $\Omega_0\xi$ in the component t^{11} and the opposite sign of $R^\mu\bar{R}^\nu$ term by an error. The author has corrected his formulas in Melrose (2008).

Integrals (17,18,19) can be evaluated knowing that

$$\int_0^\infty d\alpha \left[\alpha^n \frac{K_2(\sqrt{\rho^2-2i\rho\alpha})}{\rho^2-2i\rho\alpha}\right] = n! i^{n+1}\frac{K_{n-1}(\rho)}{\rho^2}, \quad (21)$$

$$\int_0^\infty d\alpha \left[\alpha^n \frac{K_3(\sqrt{\rho^2-2i\rho\alpha})}{(\rho^2-2i\rho\alpha)^{3/2}}\right] = n! i^{n+1}\frac{K_{n-2}(\rho)}{\rho^3}. \quad (22)$$

2.4. *Components in high-frequency limit*

I substitute the high-frequency expansions (14,15,16) into the expression (11) for the projection of the dielectric tensor $\varepsilon^\mu{}_\nu$ with the projection of the response tensor $\alpha^\mu{}_\nu$ (5) and take the integrals (17,18,19) analytically. The components of the dielectric tensor (11) in the lowest orders in Ω_0/ω are then

$$\varepsilon^1{}_1 = 1 - \frac{\omega_p^2}{\omega^2}\left(\frac{K_1(\rho)}{K_2(\rho)}\left(1+\frac{\Omega_0^2}{\omega^2}\cos^2\theta\right) + \frac{\Omega_0^2\sin^2\theta}{\omega^2\rho}\right), \quad (23)$$

$$\varepsilon^2{}_2 = 1 - \frac{\omega_p^2}{\omega^2}\left(\frac{K_1(\rho)}{K_2(\rho)}\left(1+\frac{\Omega_0^2}{\omega^2}\right) + \frac{7\Omega_0^2\sin^2\theta}{\omega^2\rho}\right), \quad (24)$$

$$\varepsilon^1{}_2 = -\varepsilon^2{}_1 = -i\eta\frac{\omega_p^2\Omega_0}{\omega^3}\frac{K_0(\rho)}{K_2(\rho)}\cos\theta, \quad (25)$$

where the plasma frequency ω_p in CGS units is

$$\omega_p^2 = \frac{4\pi n q^2}{m}. \quad (26)$$

The results reproduce the non-relativistic limits for $\rho \to +\infty$:

$$\varepsilon^1{}_1 = 1 - \frac{\omega_p^2}{\omega^2}\left(1+\frac{\Omega_0^2}{\omega^2}\cos^2\theta\right), \quad (27)$$

$$\varepsilon^2{}_2 = 1 - \frac{\omega_p^2}{\omega^2}\left(1+\frac{\Omega_0^2}{\omega^2}\right), \quad (28)$$

$$\varepsilon^1{}_2 = -\varepsilon^2{}_1 = -i\eta\frac{\omega_p^2\Omega_0}{\omega^3}\cos\theta, \quad (29)$$

where all Bessel functions of ρ approach unity[2] (Landau & Lifshits 1980; Trubnikov 1996; Swanson 2003; Bellan 2006). The corresponding relativistic limits $\rho \to 0$ of the same components are

$$\varepsilon^1{}_1 = 1 - \frac{\omega_p^2}{\omega^2}\left(\frac{1}{2T}\left(1+\frac{\Omega_0^2}{\omega^2}\cos^2\theta\right) + T\frac{\Omega_0^2\sin^2\theta}{\omega^2}\right), \quad (30)$$

$$\varepsilon^2{}_2 = 1 - \frac{\omega_p^2}{\omega^2}\left(\frac{1}{2T}\left(1+\frac{\Omega_0^2}{\omega^2}\right) + T\frac{7\Omega_0^2\sin^2\theta}{\omega^2}\right), \quad (31)$$

$$\varepsilon^1{}_2 = -\varepsilon^2{}_1 = -i\eta\frac{\omega_p^2\Omega_0}{\omega^3}\frac{\ln(T)}{2T^2}\cos\theta, \quad (32)$$

consistent with Melrose (1997c); Quataert & Gruzinov (2000)[3]. The ultra-relativistic non-magnetized dispersion relation then reads

$$\omega^2 = \frac{\omega_p^2}{2T} + c^2k^2 = \frac{2\pi n q^2}{mT} + c^2k^2 \quad (33)$$

according to the relation (12). The expression (33) is consistent with Landau & Lifshits (1980), chapter 32.

The plasma propagation effects can usually be described in terms of only the difference of the diagonal components and the non-diagonal component of $\varepsilon^\mu{}_\nu$. I define $\mathbf{X}$ to be a vector of T, θ, Ω_0/ω. I introduce the multipliers $f(\mathbf{X})$ and $g(\mathbf{X})$ to correct the expressions, when the high-frequency limit breaks. I write the difference between the diagonal components with a multiplier $f(\mathbf{X})$ as

$$\varepsilon^1{}_1 - \varepsilon^2{}_2 = f(\mathbf{X})\frac{\omega_p^2\Omega_0^2}{\omega^4}\left(\frac{K_1(T^{-1})}{K_2(T^{-1})} + 6T\right)\sin^2\theta \quad (34)$$

and the non-diagonal component with a multiplier $g(\mathbf{X})$ as

$$\varepsilon^1{}_2 = -i\eta g(\mathbf{X})\frac{\omega_p^2\Omega_0}{\omega^3}\frac{K_0(T^{-1})}{K_2(T^{-1})}\cos\theta. \quad (35)$$

Both multipliers equal unity in the high-frequency limit $f(\mathbf{X}) = g(\mathbf{X}) = 1$. Now we can turn to a more general case.

2.5. *Fitting formulas for higher temperatures*

The ultra-relativistic expressions (30,31,32) allow me to trace the T-factors in front of the first 3 expansion coefficients of the dielectric tensor in β. The coefficient at β^2 is $\sim T^3/\ln(T)$ times larger than at β. Thus at temperature $T \gtrsim 10$ the 2-nd order becomes larger than the 1-st order for the ratio $\Omega_0/\omega \sim 10^{-3}$. This indicates that the expansion in β may become invalid at these plasma parameters[4]. The multipliers $f(\mathbf{X})$ and $g(\mathbf{X})$ are likely to be far from 1. I consider only the real parts of these multipliers, since the imaginary parts correspond to absorption. The contour plots of the numerically calculated $f(\mathbf{X})$ and $g(\mathbf{X})$ for somewhat arbitrary $\theta = \pi/4$ are shown on Figure 2 and Figure 3, respectively.

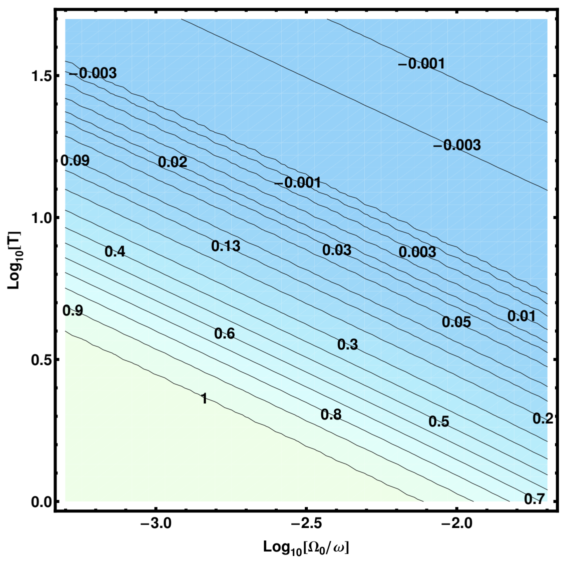

FIG. 2.— Multiplier $f(\mathbf{X})$ for the difference of the diagonal components $\varepsilon^1{}_1 - \varepsilon^2{}_2$ for $\theta = \pi/4$.

[2] The non-diagonal term has a wrong sign in Melrose (1997c).

[3] The diagonal plasma response is 2 times larger in Melrose (1997c) by an error.

[4] One cannot claim that the diagonal magnetized terms become larger then the non-diagonal (Melrose 1997c).

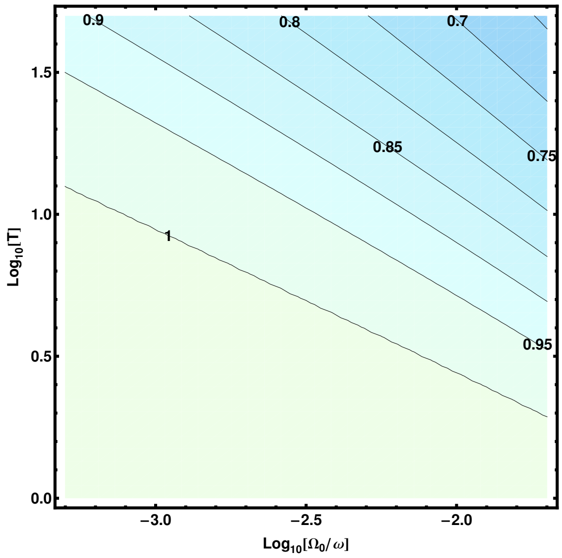

FIG. 3.— Multiplier $g(\mathbf{X})$ for the non-diagonal component $\varepsilon^1{}_2$ for $\theta = \pi/4$.

Let me define X to be the following combination of the parameters

$$X = T\sqrt{\sqrt{2}\sin\theta\left(10^3\frac{\Omega_0}{\omega}\right)}. \quad (36)$$

For the fiducial $\Omega_0/\omega = 10^{-3}$, $\theta = \pi/4$ the parameter X is just temperature $X = T$.

I first identify the boundaries, where the high-frequency limit is valid. Then I find a fit for the multipliers at higher X. The expression (34) for the difference $\varepsilon^1{}_1 - \varepsilon^2{}_2$ is accurate within 10% for $X < 0.1$ if we set $f(X) = 1$. The expression (35) for $\varepsilon^1{}_2$ is accurate within 10% for $X < 30$ if we set $g(X) = 1$. The accuracy depends on the parameter X rather than on the individual parameters T, Ω_0/ω, θ. The expression

$$f(X) = 2.011\exp\left(-\frac{X^{1.035}}{4.7}\right) - $$
$$-\cos\left(\frac{X}{2}\right)\exp\left(-\frac{X^{1.2}}{2.73}\right) - 0.011\exp\left(-\frac{X}{47.2}\right) \quad (37)$$

extends the applicability domain of the formula (34) up to $X \sim 200$. Figure 4 shows the fit for $f(X)$ in comparison with the numerical results. The expression

$$g(X) = 1 - 0.11\ln(1 + 0.035X) \quad (38)$$

extends up to $X \sim 200$ the domain of the formula (35). Figure 5 shows the fit for $g(X)$ in comparison with the numerical results.

2.6. *Exact plasma response*

The expression for the response tensor (5) is written for a vacuum wave with $|\mathbf{k}|c = \omega$. In the real plasma, the wave is modified by the plasma response. A more general self-consistent response tensor should be used (Trubnikov 1958; Melrose 1997c). One needs to solve a dispersion relation similar to the relation (12) to obtain the eigenmodes. Thus the eigenmodes and the response tensor should be computed self-consistently. One should not forget about the antihermitian and longitudinal components of the dielectric tensor $\varepsilon^\mu{}_\nu$ that modify the dispersion relation.

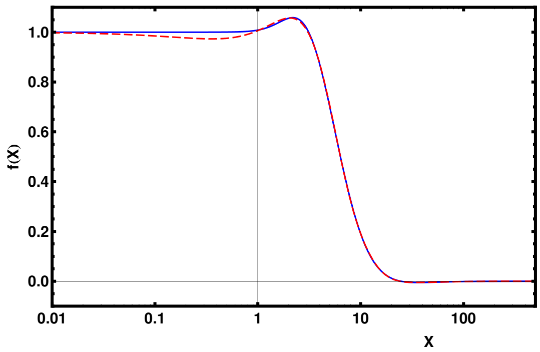

FIG. 4.— Multiplier $f(X)$ for the difference of the diagonal components $\varepsilon^1{}_1 - \varepsilon^2{}_2$. Dashed line — fitting formula (37).

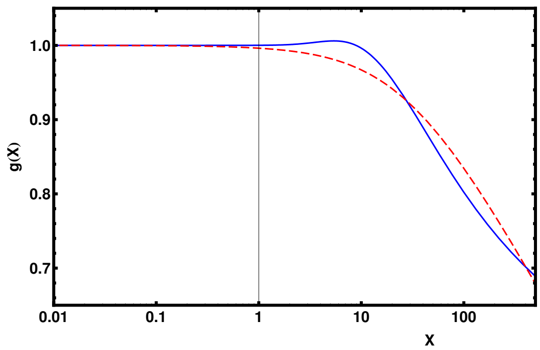

FIG. 5.— Multiplier $g(X)$ for the non-diagonal component $\varepsilon^1{}_2$. Dashed line — fitting formula (38).

2.7. *Eigenmodes*

The above calculation is applicable also to a non-magnetized plasma. Dispersion relation of EM waves in a non-magnetized plasma reads

$$\omega^2 = k^2c^2 + \omega_p^2\frac{K_1(T^{-1})}{K_2(T^{-1})} \quad (39)$$

in a high-frequency approximation $\omega \gg \omega_p$. The opposite limit of $kc \ll \omega$ was considered by Bergman (2001).

Now we turn to the magnetized case. Melrose (1997c) only considered the first terms of in the expansion of $\alpha^\mu{}_\nu$ in β to get the eigenmodes. I do the next step: consider the full expression in β in the low-density regime $kc = \omega$, but consider only the hermitian part of $\alpha^\mu{}_\nu$ in computations. The ellipticity $\Upsilon = (\varepsilon^1{}_1 - \varepsilon^2{}_2) : |\varepsilon^1{}_2|$ determines the type of eigenmodes. If $|\Upsilon| \gg 1$, then the eigenmodes are linearly polarized unless θ is close to 0. If $|\Upsilon| \ll 1$, then the eigenmodes are circularly polarized for θ far from $\pi/2$. Let me consider the fiducial model with $\Omega_0/\omega = 10^{-3}$ and $\theta = \pi/4$. Figure 6 shows the ratio Υ calculated in a high-frequency approximation (see § 2.3) (dashed line) and in a general low-density approximation (see § 2.5) (solid line). The high-frequency approximation produces the linear eigenmodes already at $T \gtrsim 10$ consistently with Melrose (1997c). However, the general low-density limit produces the eigenmodes with $\Upsilon \sim 1$ up to very high temperatures $T \sim 50$. Unexpectedly, the sign of the diagonal difference $(\varepsilon^1{}_1 - \varepsilon^2{}_2)$ changes at about $T \approx 25$.

3. APPLICATIONS

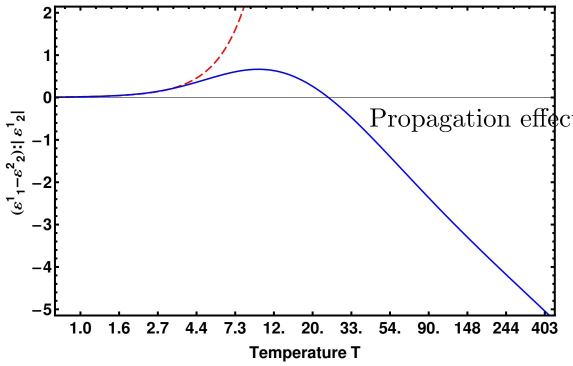

FIG. 6.— Ellipticity $\Upsilon = (\varepsilon^1{}_1 - \varepsilon^2{}_2) : |\varepsilon^1{}_2|$ of eigenmodes. The absolute value of the ratio Υ much above unity — linear eigenmodes, much below unity — circular eigenmodes. Solid line — this paper, dashed line — previous calculations.

The calculated transrelativistic propagation effects have far-reaching consequences in many topics of astronomy. Let me concentrate on four applications: propagation delay, Faraday rotation measure of light from the Galactic Center (GC), circularly polarized light from the GC, diagnostics of jets.

3.1. *Dispersion measure*

Propagation delay is an important effect in pulsar dispersion (Phillips & Wolszczan 1992). The relativistic part of this delay can be obtained from the dispersion relation (39). I retain only the first-order correction in T, since $T \ll 1$ in the interstellar medium (Cox & Reynolds 1987). Since $K_1(T^{-1})/K_2(T^{-1}) \approx 1 - 3T/2$ at low T, the non-relativistic Dispersion Measure (DM) should be modified as

$$DM_{\rm rel} = DM_{\rm nonrel}\left(1 - \frac{3}{2}T\right). \qquad (40)$$

This shows that the gas density is slightly underestimated, if the non-relativistic formulas are used[5]. However, the relativistic correction to the DM is small and can be neglected in most practical cases when $T \ll 1$. The effects in magnetized plasma are also relevant for pulsars.

3.2. *Magnetized radiative transfer*

3.2.1. *General formulae*

Relativistic plasmas exhibit a generalized Faraday rotation for a general orientation of the magnetic field (Azzam & Bashara 1987). One can decompose it into two effects: Faraday rotation and Faraday conversion. The former operates alone at $\theta = 0, \pi$, the latter operates alone at $\theta = \pi/2$, and both should be considered together for the intermediate angles. The transfer equations (Mueller calculus) for the Stokes parameters I, Q, U, V were devised to treat together the propagation effects, emission, and absorption (Azzam & Bashara 1987; Melrose & McPhedran 1991). Good approximations for emission and absorption have been long known (Trubnikov 1958; Rybicki & Lightman 1967; Melrose & McPhedran 1991; Wolfe & Melia 2006). Now one can combine them with the proper approximations of the propagation effects given by

$$\frac{d}{ds}\begin{pmatrix} I\\ Q\\ U\\ V\end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & -\rho_V & \rho_U\\ 0 & \rho_V & 0 & -\rho_Q\\ 0 & -\rho_U & \rho_Q & 0\end{pmatrix}\begin{pmatrix} I\\ Q\\ U\\ V\end{pmatrix}, \qquad (41)$$

[5] The formula in Phillips & Wolszczan (1992) has no references/checks and is not correct.

$$\rho_V = -\frac{\omega}{c} i\varepsilon^1{}_2, \quad \rho_Q = -\frac{\omega}{2c}(\varepsilon^1{}_1 - \varepsilon^2{}_2), \quad \rho_U = 0, \quad (42)$$

and do the radiative transfer calculations. Here $\varepsilon^\mu{}_\nu$ stands for the Hermitean part given by the relations (37,38) with the real multipliers $f(X)$ and $g(X)$. One of the most interesting objects for such calculations is our Galactic Center Sgr A*.

The transfer equations were recently solved for a simple time-independent dynamical model of the GC accretion (Huang et al. 2008). The authors treat the ordinary and extraordinary modes as linearly polarized. They assume these eigenmodes constitute a basis, where either U or Q components of emissivity and propagation coefficients vanish. Actually, U components vanish ($\rho_U = 0$) already in the basis $(\mathbf{e}^1, \mathbf{e}^2)$, since the projection of the magnetic field onto $(\mathbf{e}^1, \mathbf{e}^2)$ is parallel to $\mathbf{e}^1$ (see Melrose & McPhedran (1991) p.184). As I have shown in the § 2.7, plasma modes are far from being linearly polarized at temperatures $T \lesssim 10$ estimated for the GC (Sharma et al. 2007). Thus, the propagation coefficients should be taken from equations (34) and (35). The Faraday conversion coefficient ρ_Q cannot be defined via emissivities and Faraday rotation coefficient ρ_V as in Huang et al. (2008). The Faraday rotation measure was calculated from a simulated accretion profile in Sharma, Quataert & Stone (2007). However, the paper considered only the Faraday rotation and did not carry out the self-consistent treatment of propagation. It is impossible to disentangle the effects of Faraday rotation and Faraday conversion in a relativistic plasma.

3.2.2. *Faraday rotation*

The crucial part of any radiative transfer is the proper transfer coefficients. It allows one to estimate the electron density near the accreting object (Quataert & Gruzinov 2000; Shcherbakov 2008). Several formulas were suggested for the temperature dependence of the component $\varepsilon^1{}_2$ responsible for Faraday rotation. These formulas were yet given for the high-frequency approximation (see § 2.3). Let me compare them with the exact temperature dependence (25) $J = K_0(T^{-1})/K_2(T^{-1})$ and its limits. The limits are $J \to 1$ as $T \to 0$ and $J \to \ln(T)/(2T^2)$ as $T \to +\infty$. The results of this comparison are shown on Figure 7.

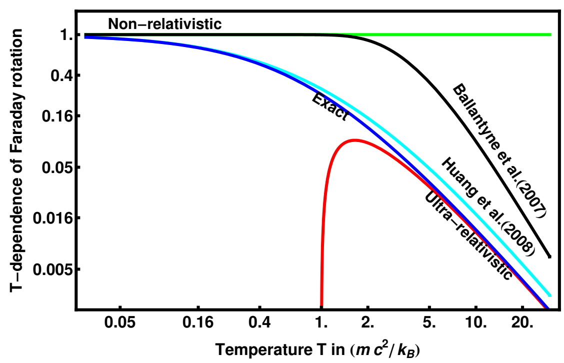

FIG. 7.— Temperature dependence of the Faraday rotation measure.

Ballantyne, Ozel & Psaltis (2007)[6] divided the thermal distribution into ultra-relativistic and non-relativistic parts as marked by the electron energy $\gamma_{\rm crit} = 10$. They sum the contributions of both species with calculated densities. To make a plot, I take their effective temperature Θ of plasma above $\gamma_{\rm crit}$ to be just temperature $\Theta = T$ and not the average kinetic energy as Ballantyne, Ozel & Psaltis (2007) suggest. This brings Θ to lower values and decreases the rotation measure. Even with this decrease the rotation measure is severely overestimated at $T \sim 1$. The convergence to the relativistic limit is not achieved even at $T \sim 30$. The paper Huang et al. (2008) found the simpler fitting formula that reproduces the limits. Their expression is quite accurate.[7]

3.2.3. *Faraday conversion*

The increase in the circular polarization of Sgr A* at frequency 1THz is predicted by Huang et al. (2008). The phase of Faraday conversion approaches unity and the destructive interference does not occur at this frequency. The result seems to be qualitatively correct regardless of the expression for the conversion measure, but the proper expressions (34) and (35) should be used for quantitative predictions.

3.2.4. *Jets*

The better treatment of propagation effects may also play a role in observations of jets. As we saw in § 2.5, the propagation effects in thermal plasma cannot be described in the lowest orders in Ω_0/ω, if the temperature T is sufficiently high. Power-law distribution of electrons can have a quite high effective temperature. Thus the high-frequency limit (Sazonov 1969; Jones & O'Dell 1977; Melrose 1997b) may not approximate well the hermitian part of the response tensor. Careful analysis of jet observations (Beckert & Falcke 2002; Wardle et al. 1998) may be needed. It should be based at least on the expressions for $\varepsilon^{\mu}{}_{\nu}$ in a general low-density regime.

[6] The paper Ballantyne, Ozel & Psaltis (2007) has likely confused the 3-D projection of the 4-D response tensor in $j^{\mu} = \alpha^{\mu\nu} A_{\nu}$ (Melrose 1997c) with the 3-D response tensor $\mathbf{j} = \alpha_{ij}\mathbf{A}$ that has the opposite sign.

[7] "Temperature" γ_c in Huang et al. (2008) should be redefined as $\gamma_c = 1 + T$, otherwise the lower limit is not reproduced.

4. DISCUSSION & CONCLUSION

This paper presents several new calculations and amends the previous calculations of propagation effects in uniform magnetized plasma with thermal particle distribution equation (1). The expression (5) for the correct response tensor is given in a high-frequency approximation. The exact temperature dependence (2.4) and (25) is found in first orders in Ω_0/ω in addition to the known highly-relativistic and non-relativistic results. The higher order terms may be important for relativistic plasmas in jets and hot accretion flows. The fitting expressions (37) and (38) are found for the dielectric tensor components (34) and (35) at relatively high temperatures.

The results of numerical computations are given only when the corresponding analytical formulas are found. One can always compute the needed coefficients numerically for every particular frequency ω, plasma frequency ω_p, cyclotron frequency Ω_0, and distribution of electrons. However, the analytic formulas offer a simpler and faster way of dealing with the radiative transfer for a non-specialist. The eigenmodes were not considered in much detail, since radiative transfer problems do not require a knowledge of eigenmodes. However the knowledge of eigenmodes is needed to compute the self-consistent response tensor (see § 2.6).

The response tensor in the form (5) can be expanded in Ω_0/ω and ω_p/ω. This expansion is of mathematical interest and will be presented in a subsequent paper as well as the expressions for a power-law electron distribution. Propagation through non-magnetized plasmas will also be considered separately.

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